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COLORING PLANAR PERFECT GRAPHS BY DECOMPOSITION

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This paper describes a decomposition scheme for coloring perfect graphs. Based on this scheme, one need only concentrate on coloring highly connected (at least 3-connected) perfect graphs. This idea is illustrated on planar perfect graphs, which yields a straightforward coloring algorithm. We suspect that, under appropriate definition, highly connected perfect graphs might possess certain regular properties that are amenable to coloring algorithms.

1. Introduction

We denote a graph G by a pair (V, E), where V (or $V(G)$) denotes the finite vertex set of G and E (or $E(G)$) denotes a set of edges connecting vertices of G. The graphs we are interested in have no loops or multiple edges and are undirected. Let $\omega(G)$ denote the maximum size of a clique in G and $\gamma(G)$ the chromatic number of G (the minimum number of independent set of vertices that cover all vertices of G). In general, we have $\omega(G)\leqq\gamma(G)$. A graph G is called *perfect* if $\omega(H)=\gamma(H)$ for every induced subgraph H of G. It is easy to see that *odd holes* (chordless odd cycles) and *odd antiholes* (complements of odd holes) are not perfect. Berge [1] conjectured that a graph is perfect if and only if it does not contain odd holes or odd antiholes as induced subgraphs (the so-called strong perfect graph conjecture). The validity of this conjecture on planar graphs has been established by Tucker [6]. His proof could be used to construct minimum colorings for planar perfect graphs [7]. However, it involved a number of detailed case analyses and is not likely to be applicable for general perfect graphs. In this paper we give a coloring algorithm based on a certain decomposition scheme. The algorithm proceeds by decomposing a given graph into components which can be colored easily and then matching up the colors for the entire graph. Our emphasis in this paper is on the algorithms' conceptual simplicity rather than on its computational efficiency. In the next section we describe a separation scheme which is valid for general perfect graphs. We then give a simple coloring algorithm on an inseparable planar perfect graph in Section 3. Finally, a number of open problems are raised in Section 4. Our approach is entirely different from the ellipsoid algorithms of Grötschel et al. [2].

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2. A decomposition scheme for coloring perfect graphs

In this section we describe a composition scheme that preserves perfection. Let G_1, G_2 be two perfect graphs. Let C_1, C_2 be two cliques of the same size in G_1, G_2 respectively. Let x_1 (resp. x_2) be a vertex in $G_1 \setminus C_1$ (resp. $G_2 \setminus C_2$) so that the number of vertices in C_1 adjacent to x_1 is equal to the number of vertices in C_2 adjacent to x_2. Let G be a graph obtained from G_1 and G_2 by indentifying x_1 with x_2 and C_1 with C_2 in such a way that a vertex in C_1 adjacent to x_1 is identified with a vertex in C_2 adjacent to x_2 (but otherwise arbitrarily). A path is said to be *even* (resp. *odd.*) if it contains an even (resp. odd) number of edges.

Our main result in this section is the following

Theorem 1. *If the graph G constructed above does not contain odd holes, then it is perfect.*

Proof. All we have to show is that $\omega(G)=\gamma(G)$ since the same argument would apply to any induced subgraph of G.

Without loss of generality, assume $\omega(G_1) \geqq \omega(G_2)$. It is clear that $\omega(G)= =\max(\omega(G_1), \omega(G_2))=\omega(G_1)=\gamma(G_1)$. We shall show that G can be colored using $\gamma(G_1)$ colors. Suppose we have colored G_1, G_2 using $\gamma(G_1)$, $\gamma(G_2)$ colors, respectively (assume the colors are numbered from 1 to $\gamma(G_1)$). We can assume that the color of a vertex y_1 in C_1 is the same as the color of the vertex y_2 in C_2 with which y_1 is to be identified in G. If, in these colorings of G_1 and G_2, x_1 receives the same color as x_2 does, then by identifying these similarly colored vertices in G_1 and G_2 we would be able to color G properly using $\omega(G)$ colors.

On the other hand, if x_1 receives a color i and x_2 receives a color j $(\neq i)$, then we claim that one of the following two color switchings can retain the same color for x_1 and x_2 without changing the colors of vertices in C_1 and C_2 and hence, G can again be colored using $\omega(G)$ colors. The color switchings are: in the subgraph induced on vertices colored i or j in G_1 (resp. G_2), switch colors i and j in the connected component B_1 (resp. B_2) containing x_1 (resp. x_2). Suppose both switchings change colors of some vertices in C_1 and in C_2. Then both colors i and j must be present in C_1 (and hence, in C_2). For otherwise, let the missing color be i and let u_1 and u_2 be vertices with color j in C_1 and C_2, respectively. We would get an even induced path connecting x_1 to u_1 and an odd induced path connecting x_2 to u_2 which together create an odd hole in G.

Hence, let u_1^i and u_1^j (resp. u_2^i and u_2^j) be vertices of C_1 (resp. C_2) colored i and j, respectively, so that u_1^i and u_2^i are identified to u^i in G and u_1^j and u_2^j are identified to u^j in G. Since x_1 is colored i and x_2 is colored j, none of u_1^i, u_1^j can be adjacent to x_1 in G_1 and none of u_2^i, u_2^j can be adjacent to x_2 in G_2. Let P_1 be a shortest path in B_1 connecting x_1 to one of u_1^i or u_1^j and P_2, a shortest path in B_2 connecting x_2 to one of u_2^i or u_2^j. If both P_1 and P_2 connect vertices with the same color in C_1 and C_2, respectively, then one path must be even and the other one odd, which would create an odd hole in G. Hence P_1 connects a vertex of C_1 whose color is different from that of the vertex connected by P_2. In this case, both P_1 and P_2 have the same parity. But then, the edges of P_1 and P_2 together with the edge (u^i, u^j) form an odd hole in G. ∎

From the proof of Theorem 1 we get a coloring scheme for a perfect graph G: if G can be obtained by identifying a clique and an additional vertex from two perfect graphs G_1 and G_2 as given above, then we can get a minimum coloring of G by matching up minimum colorings on G_1 and G_2 as described in the proof. The existence of such G_1 and G_2 can easily be recognized by searching for a clique and an additional vertex (either the clique or the vertex could be empty) in G whose deletion disconnects the graph; in case this clique-vertex cutset exists we call G *2-separable;* otherwise *2-inseparable.* Notice that 2-inseparable graphs with 3 or more vertices are necessarily 3-connected since we can let C_1, C_2 be either empty sets or single vertex sets. In view of the recent generalization of "2-connected" perfect graphs [3], the cliques used in the 2-separation can be extended to more general subgraphs.

It is not difficult to show that the number $\tau(G)$ of subgraphs which can be generated from a given graph G by this decomposition is bounded by $O(|V(G)|)^3$. If G is Q-inseparable then $\tau(G)=1$. Otherwise we can find G_1, G_2 which decompose G (as described above). Let K_i (resp. K) be a clique of maximum cardinality of G_i (resp. G), and $a_i=|V(G_i)|-|K_i|$, $a=|V(G)|-|K|$. Clearly, $|K|=\max(|K_1|, |K_2|)$. Let C be the clique of G identified from C_1 and C_2. Then

$$\begin{aligned} a_1+a_2 &= |V(G_1)|+|V(G_2)|-|K|-\min(|K_1|, |K_2|) \\ &\leqq |V(G_1)|+|V(G_2)|-|C|-1-|K|+1 = |V(G)|-|K|+1 \leqq a+1. \end{aligned}$$

Next we claim that $\tau(G)\leqq|V(G)|\,(a+1)^2$ by induction on $|V(G)|$. This is obviously true if $|V(G)|=1$ or $|V(G)|=2$. Hence assume $|V(G)|\geqq 3$. Therefore,

$$\begin{aligned} \tau(G) &\leqq 1+\tau(G_1)+\tau(G_2) \leqq 1+|V(G_1)|(a_1^2+1)+|V(G_2)|(a_2^2+1) \\ &\leqq |V(G)|(a_1^2+a_2^2+2) \leqq |V(G)|(a+1)^2. \end{aligned}$$

Since $a<|V(G)|$, we have $\tau(G)\leqq|V(G)|^3$.

For the class of planar perfect graphs, we can use a modified scheme which guarantees that the number of subgraphs generated is $O(|V(G)|)$. Given a connected planar perfect graph with $\omega(G)=3$, delete all edges of G which are not contained in any triangle. Then color the remaining graph G' using three colors (through decomposition), put these edges back in one by one and adjust the colors accordingly. This preprocessing step is justified by

Lemma 1. *Consider an edge e in a perfect graph G with $\omega(G)>2$. If e is not contained in any triangle, then $G\setminus e$ is also perfect and $\gamma(G)=\gamma(G\setminus e)$.*

Proof. It is clear that $\omega(G)=\omega(G\setminus e)$ and hence $\omega(H)=\omega(H\setminus e)$ for every induced graph of G. Therefore $G\setminus e$ is perfect and $\gamma(G)=\gamma(G\setminus e)$. ∎

Algorithmically, once a coloring for $G\setminus e$ is available, we can obtain a coloring for G by coloring the two end vertices of e differently through a color switching if necessary. Therefore, to complete our discussion, all we have to demonstrate is that the number of 2-inseparable component graphs generated from G' is $O(|V(G)|)$. This can be justified by the following observations. Every such component graph contains at least one triangle, every triangle of G' can be in at most two component graphs (when this triangle is used as a clique cutset, it is duplicated in two subgraphs) and the number of triangles is $O(|V(G')|)$.

The search for a cutset in a subgraph H takes at most $O(|V(H)|^2)$ time. Hence the total decomposition process for G' takes at most $O(|V(G')|^3)$ time.

3. Coloring 2-inseparable planar perfect graphs

Consider a planar perfect graph G which is 2-inseparable. Consider any plane imbedding of G. If G is bipartite, it can be 2-colored. If G is a 4-clique it can be 4-colored. Otherwise, G cannot contain any 4-clique and we can show that G is an induced subgraph of a uniquely 3-colorable planar graph $\tilde{G}$ which is constructed as follows. For each face of G which is not a triangle, add a vertex in its interior and connect this vertex to every vertex on the boundary of this face. $\tilde{G}$ is clearly a plane graph and every face of $\tilde{G}$ is a triangle. If we can show that every vertex of $\tilde{G}$ has an even degree (i.e., $\tilde{G}$ is Eulerian, its dual is bipartite), then by Kempe's Theorem (e.g. see Theorem 19D of [8]), $\tilde{G}$ is 3-colorable. The actual coloring procedure is trivial since the colors of a single triangle can be uniquely extended to the entire graph.

Our main result in this section is the following

Theorem 2. *Let G be a 2-inseparable plane perfect graph that contains 3-cliques but no 4-cliques. For any vertex x of G, let C_x be the boundary of the face created by deleting x from G. Then C_x must be an induced even cycle.*

Proof. Suppose not. Then there must exist some edge, say (y, z), connecting two non-consecutive vertices y and z on the cycle C_x. But then, the 2-clique $\{y, z\}$ and vertex x constitute a 2-cutset for G. ∎

Let $u_1\ldots u_{i_1}v_1\ldots v_{j_1}u_{i_1+1}\ldots u_{i_2}v_{j_1+1}\ldots v_{j_2}\ldots u_{i_{r-1}+1}\ldots u_{i_r}u_{j_{r-1}+1}\ldots v_{j_r}$ be a consecutive list of vertices in the cycle C_x, where each of the u's is connected to x but none of the v's is. By Theorem 2, C_x is an induced even cycle. Thus, i_r+j_r is even. Now each cycle $xu_{i_l}v_{j_{l-1}+1}\ldots v_{j_l}u_{i_l+1}$ is an induced cycle $l=1, \ldots, r$ (let $j_0=0$, $u_{i_r+1}=u_1$) and hence, must have even length. Therefore, j_l-j_{l-1} is odd for every $l=1, \ldots, r$. For each face enclosed by a cycle $xu_{i_l}v_{j_{l-1}+1}\ldots v_{j_l}u_{i_l+1}$, $l=1, \ldots, r$ we add a vertex in G and connect it to x. Hence the degree of x in $\tilde{G}$ is

$$i_r+r = i_r+\sum_{l=1}^{r} 1 = i_r+\sum_{l=1}^{r}[(j_l-j_{l-1})-(j_l-j_{l-1}-1)]$$

$$= i_r+j_r-\sum_{l=1}^{r}(j_l-j_{l-1}-1)$$

which is an even number since i_r+j_r is even and $j_l-j_{l-1}-1$ is even for every $l=1, \ldots, r$.

Since the boundary of each face of G is an induced cycle, by Theorem 2, it must be even. Hence each added vertex of $\tilde{G}$ has an even degree. Therefore $\tilde{G}$ is indeed Eulerian and uniquely 3-colorable.

Thus, coloring a 2-inseparable component graph takes linear time. Therefore, the overall complexity of coloring a planar perfect graph G is dominated by the decomposition step which takes at most $O(|V(G)|^3)$ time.

4. Conclusion

In this paper we presented a novel algorithm for coloring perfect graphs. As we have shown, the general coloring problem for perfect graphs can be reduced to coloring perfect graphs which are highly connected (at least 3-connected). It has been demonstrated that 2-inseparable planar perfect graphs possess a certain property which enables us to color them easily. We suspect that a more general notion of inseparability on perfect graphs can lead to similar results for general perfect graphs.

Based on a different decomposition scheme, the author has developed a recognition algorithm as well as a packing algorithm for the class of planar perfect graphs [4, 5].

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